OUR MUSICAL IDIOM

BY

ERNST LECHER BACON

WITH AN INTRODUCTION

BY

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INTRODUCTION.

The effort to expand the means of musical expression is as old as the art itself. It is recorded in each chapter of musical history; it has been interrupted only during those periods of the art's development wherein the composer has been concerned with the completion of art-types already defined.

Every advance in the art has been prefaced by a period of experimental effort, which has sought new modes of expression. So soon as these modes of expression have been defined and their tendencies and the laws governing them have been apprehended, experiment has been replaced by careful conformance to law and tradition, which has operated to the perfection of the new art type.

The present is preeminently an epoch of experimentation. The old art types have been completed. The harmonic vocabulary based upon the sequences of tonality established in these completed art forms also has been exhausted and for the past half century composers have been concerned with the development of new harmonic idioms. (Viz., Liszt, Wagner, Strauss, Franck, D'Indy, Debussy, Ravel, Shoenberg, Busoni.) As these composers have discovered and employed new harmonies, new scales and new sequences of tonality with their resultant new harmonic progressions, the theoretician has endeavored to classify their discoveries according to his established system with results weirdly confusing. The crying need of the moment seems to be a new system for the naming and classifying of all possible tonal combinations.

"Harmony is that which sounds together," wrote Bernard Ziehn twenty-five years ago. But the average theoretician comprehends only those simultaneously produced sounds which may be arranged in series of superimposed thirds. In the meantime the composer has consciously employed many harmonies which are not formed of superimposed thirds. (Viz., Debussy's major second as the first interval of the tonic chord, or Shoenberg's combinations of superimposed fourths, to cite familiar examples.) The executive artist, upon whom the composer is dependent for the delivery of his message, is, in turn, dependent upon the theoretician for a logical classification of the new harmonies. The composer of the present is almost equally dependent upon some scientific classification of his material. Naturally the public has first looked to him for this classification. But he seems able to make it only for himself, as Reger and Shoenberg have done. In any event the world has been slow to adopt these special classifications and is still seeking a general system that will include all possible harmonies in logical order.

That system has been evolved by Mr. Ernst Lecher Bacon of Chicago, who by applying the principles of algebraic permutations to the problem has succeeded in formulating all harmonies that may possibly exist in the present system of twelve tones (of itself a most important service) and having formulated them, has found a system of nomenclature which actually describes any possible combination of tones and makes a general or special classification possible.

The value of this new system of nomenclature to the executive artist is immediately apparent. That puzzled individual may name and classify the new tonal combinations which he is required to memorize and present convincingly to the public. The composer is even more importantly served by Mr. Bacon's researches. For he is shown at a glance all possible harmonies (there are but 350) and all possible scales (of which there are about 1490). He may select from the clear and concise tables placed at his disposal those harmonies and scales which seem to him useful and beautiful and having familiarized himself with their color and feeling, in short, made them a part of his own consciousness, may employ them subjectively to the expression of feeling and sensibility, to the building up of his own especial harmonic idiom. For though the composer's work is best when it is most subjective, he is constantly obliged to concern himself with the facts of his art and out of these facts to fashion that delicate fabric of feeling and fantasy which is to give freer and fuller powers of expression to the music of the future.

GLENN DILLARD GUNN.

THE FORMATION OF SCALES.

The chromatic scale has become established as the basis of modern harmony. Though the major and minor modes are still accorded that recognition which the printed key signature seems to imply, modern compositions so bristle with accidentals that even to the eye, and still more to the listening ear, is it evident that the restrictions of the major and minor system have been destroyed.

However, few of our modern composers treat the chromatic scale purely in itself; the intrusion of other scales which are synthetically formed from the chromatic scale is always felt. The chromatic scale is only the analytic product of others, which consist of combinations of its semitones, wherein intervals are found which are collections of semitones. Of these synthetically formed scales there are many in number but few in use; and each may form a separate harmonic basis. Beethoven and Liszt, the latter more notably, occasionally used scales differing markedly from the major and minor; but their appearance was only incidental, and the scales were rarely made use of as bases for harmonic systems. Busoni created, as he says, 113 different scales through rearrangements or permutations of the intervals of the major and minor scales.[1] Debussy has used a few unfamiliar scales, notably the whole tone scale, for a thorough harmonic basis. However, as will be shown, a vast number of scales that have never before been conceived are opened to discovery through the application of the principles of algebraic permutation to arrangements of tonal sequences.

A *scale* is a series of ascending or descending tones. Such a series may conform to a pattern, a regularly recurring succession of intervals in certain order, bounded by a fixed interval; or it may not conform to pattern. The pattern may or may not conform to the duodecimal system. If the scale conform to pattern it must be bounded by a fixed interval. Intervals of simple physical ratio are preferred, and these are found, slightly tempered, in the duodecimal system.

Until now, the octave only has been consciously used as a fixed

[1] His figures are incorrect as, mathematically computed, the number of permutations of the combination of intervals is: (*a*) of the major scale, 21; (*b*) of the minor scale, 140.

interval, but there is no reason why, in specialized cases, other intervals could not exist between corresponding tones in recurrences of the pattern. We may have scales repeating at each fourth, fifth, sixth, seventh, ninth, tenth, etc. But because we are at present engaged in a classification of scales which incidentally involves the discovery of a multitude of unheard-of ones, and because a classification of such scales as these would be of formidable length, we must be content to study that most important class of scales in which each succeeding repetition begins an octave above or below the preceding one.

Again we must distinguish between two classes of scales whose basis is the octave. The first class is that one in which the smallest scale-units in the octave number 12; this is our *duodecimal* system. The second class contains many systems, in each of which the number of the smallest units is either greater or less than 12. We will consider both of these classes, for in the consideration of the first class we can enlarge considerably the present scope of the duodecimal system, while in the consideration of the second class we may discern dimly certain possibilities of the future. First we will discuss the scale possibilities of the duodecimal system.

By a division of the octave into twelve parts the common *chromatic* scale is formed. Now by grouping together certain of these twelfths of an octave, the so-called "semitones," we may form scales whose gradation is uneven and less refined than that of the chromatic scale. If we are given a certain combination of intervals which, added together, give the octave, we can permutate these in a number of different ways; that is, we can rearrange the given intervals to form different scales. We also may have combinations in which the same intervals occur more than once. If n is the number of intervals between octaves in the scale and n_1 of them are alike, and n_2 others are alike, etc., the number of scales (P) that can be formed by permuting the given combination of intervals is:

$$P = n!/n_1!n_2!\ldots\ldots$$

(The exclamation point, read "factorial," denotes that the number which it follows is a product of all integers less than and including itself, each integer being a factor only once.)

For example, we desire to find the number of scales that can be formed with the intervals of the major scale. The major scale consists of 5 whole tones and 2 half tones, making a total of 7 intervals.

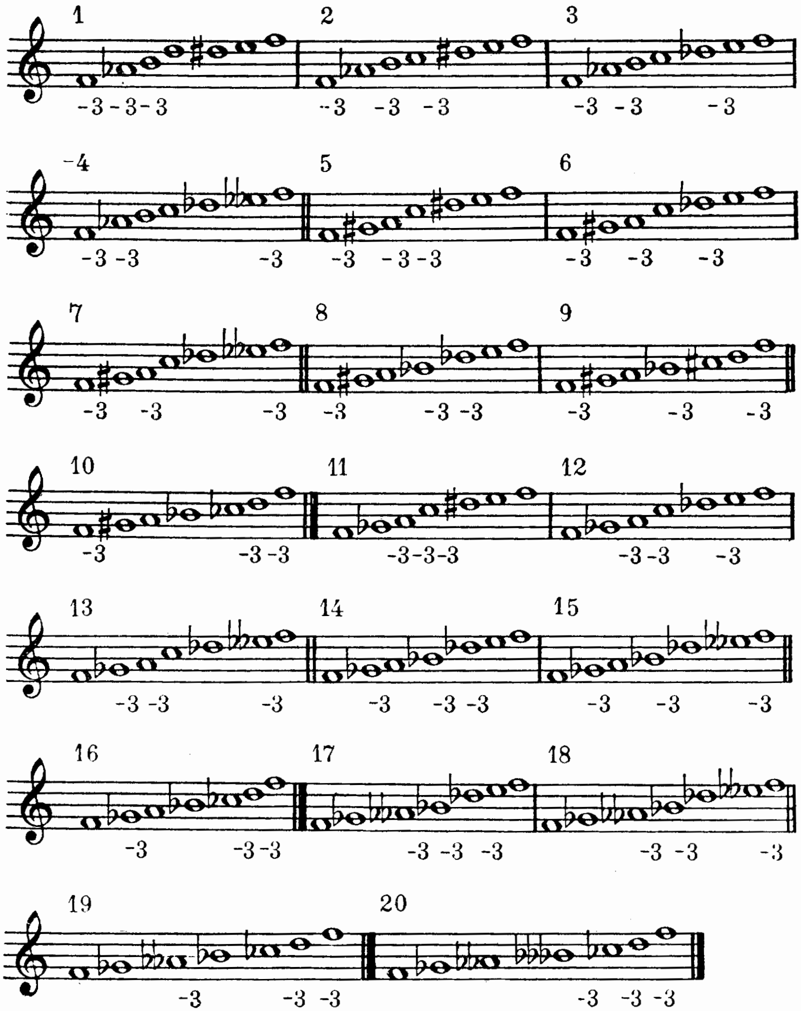
Thus $P = (n!/n_1!\,n_2!) = (7!/5!.2!)$

$$= [1.2.3.4.5.6.7/(1.2.3.4.5.)\ (1.2)]$$

$P = 42/2 = 21.$

As an example of the way in which all possible scales can be formed out of a certain combination of intervals, 20 scales will be formed out of the combination (three minor thirds and three minor seconds) as follows:

The symbol (— 3) will be written below respective minor thirds.



It will be observed that a new series begins with each double or triple bar.

A triple bar is written before each chromatic elevation of the lowest minor third.

A double bar is written before each chromatic elevation of the middle minor third.

The uppermost minor third always starts at its lowest possible position and is raised successively to its highest possible one, after which a change is made in the relative position of the lower minor thirds.

This method of forming all scales from a certain combination of intervals is purely arbitrary.

Now it is possible to form a great number of combinations in which the sum of the intervals is an octave. Moreover, as we have seen, usually a number of scales can be formed out of each combination. Each scale is to be considered a permutation of the combination's intervals.

In the following table will be found every combination possible with intervals as small as the minor second and not greater than the major third. Intervals larger than the major third are not used because in the formation of scales they would make gradation too abrupt and uneven. The table will also include calculations of the number of permutations (to be regarded as the number of scales) possible with each respective combination, according to the formula. The vertical columns contain the intervals minor 2d, major 2d, minor 3d, major 3d, respectively. The horizontal rows of numbers are the combinations. A number (n) falling in a vertical column (v) means that the interval (v) is repeated n times in the combination in which n lies (see Table I).

To make the function and construction of the table more plain two of the combinations may be explained. Combination 1 indicated by the number in the extreme left-hand column, consists of twelve semitones. It is therefore a formula of the chromatic scale, and has therefore only one permutation. Combination 21 contains two minor seconds, two major seconds and two minor thirds. From it may be formed fifteen scales or permutations.

Means have now been shown to find all possible scales in the twelve-tone system, scales which have intervals exceeding the major third in size being omitted. Adding the number of permutations formed with all combinations a total of 1490 scales is found.

A systematic study of these 1490 new scales would lead to the discovery of many valuable scales. I have found many that are interesting by this method, but will mention only a certain class of these scales, which I will call *equipartite* for want of a better name.

TABLE I.

	Combinations				Permutations			Combinations				Permutations	
	Minor Seconds	Major Seconds	Minor Thirds	Major Thirds	Calculations	Permutations		Minor Seconds	Major Seconds	Minor Thirds	Major Thirds	Calculations	Permutations
1	12				P12!/12!	1	18	3		3		6!/3! 3!	20
2	10	1			11!/10!	11	19	2	5			7!/2! 5!	21
3	9		1		10!/9!	10	20	2	3		1	6!/2! 3!	60
4	8	2			10!/2! 8!	45	21	2	2	2		6!/2! 2! 2!	90
5	8			1	9!/8!	9	22	2	1		2	5!/2! 2!	30
6	7	1	1		9!/7!	72	23	2		2	1	5!/2! 2!	30
7	6	3			9!/3! 6!	84	24	1	4	1		6!/4!	30
8	6	1		1	8!/6!	56	25	1	2	1	1	5!/2!	60
9	6		2		8!/2! 6!	28	26	1	1	3		5!/3!	20
10	5	2	1		8!/2! 5!	168	27	1		1	2	4!/2!	12
11	5		1	1	7!/5!	42	28		6			6!/6!	1
12	4	4			8!/4! 4!	70	29		4		1	5!/4!	5
13	4	2		1	7!/2! 4!	105	30		3	2		5!/2! 3!	10
14	4	1	2		7!/2! 4!	105	31		2		2	4!/2! 2!	6
15	4			2	6!/2! 4!	15	32		1	2	1	4!/2!	12
16	3	3	1		7!/3! 3!	140	33			4		4!/4!	1
17	3	1	1	1	6!/3!	120	34				3	3!/3!	1
												Total	1490

An *equipartite* scale is one in which the same pattern of intervals is repeated an integral number of times within the octave. If a scale is *bipartite* a group of intervals will appear twice within the octave with no remainder; if the scale is to begin on F its two parts begin, respectively, on F and B. As a result in this case it is immaterial whether the tonic is B or F, for the scale sounds alike either way, except for the transposition.

We may split the sum of twelve semitones (semitones being regarded as intervals) into two parts or three. Dividing it into two parts, each part containing six semitones, allows us again to divide this semi-octave into two or three parts. Dividing the octave into

TABLE II (FOR BIPARTITE SCALES).

Combinations					Permutations	
No.	1 Minor Second	2 Major Second	3 Minor Third	4 Major Third	Calculations	No. of Perm.
1	6				P=6!/6!	1
2	4	1			P=5!/4!	5
3	3		1		P=4!/3!	4
4	2	2			P=4!/2! 2!	6
5	2			1	P=3!/2!	3
6	1	1	1		P=3!	6
7		3			P=3!/3!	1
8		1		1	P=2!	2
9			2		P=2!/2!	1
					Total	29

TABLE III (FOR TRIPARTITE SCALES).

Combinations					Permutations	
No.	1 Minor Second	2 Major Second	3 Minor Third	4 Major Third	Calculations	No. of Perm.
1	4				P=4!/4!	1
2	2	1			P=3!/2!	3
3	1		1		P=2!	2
4		2			P=2!/2!	1
5				1	P=1!	1
					Total	8

three parts, each part has four semitones, which may again be divided by two. Thus we may split the octave into 2, 3, 4, and 6 equal parts. Scales formed by such divisions may be called, respectively, bipartite, tripartite, quadripartite, and sexpartite. As the last two types may be classed under the first and second they do not require a separate classification. In Tables II and III the combinations in the bipartite and tripartite types are given; in other words, the possibilities of combinations with six and four semitones, respectively, are shown. Each arrangement of a combination is then repeated in the remaining half or two-thirds of the octave.

A few interesting equipartite scales are herewith shown:

(1, 2 and 3 are from Table 2, combination No. 4; 4, 5 and 6 are from Table 3, combination No. 3.)

Scales formed by permutating combination No. 4 in Table II, and combinations Nos. 2 and 3 in Table III are especially interesting. No. 1 is formed by alternating major and minor seconds, while No. 3 is formed in the same way, except that in it the order of the intervals of No. 1 is reversed. Even such a mechanically formed scale as this sounds beautiful and original. It is a noteworthy fact that in scales 1 and 3 the chords formed on every degree are diminished. Scales Nos. 4 and 5 are built similarly; only a minor third and a minor second alternate. Chords formed on every degree of these scales are augmented.

SCALES FORMED FROM SYSTEMS OTHER THAN THE DUODECIMAL.

Although to-day the importance of systems containing other intervals than multiples of semitones is questionable, it is nevertheless interesting to know that such systems may be exploited for scale and harmonic possibilities in the same manner as our present system. Busoni has already experimented with the *tripartite* tone scale; that is, a scale in which each whole tone is divided into three instead of two whole parts. The physicist may scorn the idea of a new system, knowing that the duodecimal system contains the simplest physical intervals, yet it must be remembered that the perfect intervals are also not found in the 12-tone system, because of "tempering." Moreover in the other systems many of the most important intervals of the duodecimal system will be duplicated. Although probably no system will ever be of equal importance with the duodecimal, it is not inconceivable that, just as certain new scales within our present system have been chosen by recent composers as harmonic and melodic idioms of expression, so certain "foreign" systems may once be chosen for similar purposes.

Accordingly, we are to consider any equal divisions of the octave. However, certain divisions, as for example into 11 or 13 equal parts, are not of importance, since the intervals formed in this way would only be confounded with poorly tuned intervals of the 12-tone scale. In order to discriminate in the selection of numbers with which to divide the octave it is well to choose only those numbers which are multiples of the smallest prime numbers, 2, 3, and 5. We may call each of these systems an "N-tone chromatic system." If the system is one in which the number of smallest intervals is 9, we may call it a 9-tone chromatic system. We are not bound to confine the use of the term "chromatic" to our duodecimal system, since in its musical application the word is used to describe a succession of the smallest possible intervals.

In considering the N-tone chromatic systems we may go through the same steps through which we have passed in considering the duodecimal system. In each of these unfamiliar systems there are chromatic intervals which may be combined and permutated to form scales of more rapid and uneven gradation. Just as before, we have to set a certain limit to the size of an interval employed in one of these scales. In the five-tone system a coupling of only two chromatic intervals produces an interval almost too great to exist in a scale of moderately refined gradation. In the 24-tonal system a coupling of as many as 6 intervals into 1 is acceptable. It will readily be seen that to construct tables for all N-tonal systems through which an infinite number of gradations is possible, would require much space. It has already been stated that those systems having numbers of chromatic intervals equal to multiples of 2, 3 and 5, are most important. They are systems of 4, 5, 6, 8, 9, 10, 12, 14, 15 16, 18, 20, 21, 22, 24, etc., chromatic tones; for demonstration I will select only 2 of these; namely 8- and 9-tone systems.

In the tables that follow I will give the number of combinations and calculate the number of permutations for each combination with the selected N-tonal systems. In other words, scale possibilities with systems having 8 and 9 tones will be shown in each respective table.

TABLE II, N=8

	Combinations			Permutations	
	$\frac{1}{8}$ Octave	$\frac{2}{8}$ Octave	$\frac{3}{8}$ Octave	Calculations	Permutations
1	8			8!/8!	1
2	6	1		7!/6!	7
3	5		1	6!/5!	6
4	4	2		6!/2! 4!	15
5	3	1	1	5!/3!	20
6	2	3		5!/2! 3!	10
7	2		2	4!/2! 2!	6
8	1	2	1	4!/2!	12
9		4		4!/4!	1
10		1	2	3!/2!	3
				Total	81

Intervals used do not exceed $\frac{3}{8}$ octave. Number of intervals corresponding to Duodecimal System=4.

TABLE III, N=9

	Combinations			Permutations	
	$\frac{1}{9}$ Octave	$\frac{2}{9}$ Octave	$\frac{3}{9}$ Octave	Calculations	Permutations
1	9			9!/9!	1
2	7	1		8!/7!	8
3	6		1	7!/6!	7
4	5	2		7!/2! 5!	21
5	4	1	1	6!/4!	30
6	3	3		6!/3! 3!	20
7	3		2	5!/2! 3!	10
8	2	2	1	5!/2! 2!	30
9	1	4		5!/4!	5
10	1	1	2	4!/2!	12
11		3	1	4!/3!	4
12			3	3!/3!	1
				Total	149

Intervals used do not exceed $\frac{3}{9}$ octave or a major third, as translated.
Number of intervals corresponding to Duodecimal System=3

The Numbers of Tones and Intervals Found Correspondingly in Any Two N-Tonal Systems.

If we choose a common tonic for all N-tone chromatic scales we will find certain other tones which are common to two or more of these scales. For example, if we form both a 9-tone chromatic and a duodecimal scale upon C, we will expect to find two tones in common besides the C and its octave. They will be E and G sharp; for each of these tones marks the partition of the octave into three equal parts. This means that certain intervals in one system are the same as intervals of another. But an interval common to two systems cannot be the same multiple of the smallest unit in each system. If we desire to find the number of intervals which are found correspondingly in each of the two systems, we need merely to find the largest factor common to the number of chromatic divisions of both systems. For example, to find the number of intervals which are common to the 18-tone and the 12-tone chromatic scales we find the G. C. F. of 18 and 12, which is 6. This is the desired number. Of course, intervals which are multiples of this common interval (the whole-tone, in this case) are also common to both systems.

Intervals of N-Tone Chromatic Scales.

Throughout our entire treatment of scale possibilities there is one interval which remains constant; namely, the octave. The ratio of this interval, that is the ratio[2] of the vibration frequency of the higher tone to that of the lower tone is always 2. If N is the frequency of the lower tone, its octave is 2 N. Now N and 2 N may be written as 2^0 N and 2^1 N respectively, since any quantity with an exponent 0 equals unity. It is evident that the frequencies of any tones between $N \times 2^0$ and $N \times 2^1$ can be expressed as N times the coefficient 2 with an exponent varying between 0 and 1.

If the octave contains *r* equal intervals, the difference between 0 and 1 of the exponent of 2 will be divided into *r* parts. This is true because (*a*) equal intervals form equal ratios of vibration; and (*b*) equal ratios may be expressed as the quotients of a constant in which the difference of the constant's exponents in the numerators and respective denominators remains constant. To illustrate:

$$2^1/2^0 = 2^{1-0} = 2$$
$$2^6/2^5 = 2^{6-5} = 2.$$

Hence $2^1/2^0 = 2^6/2^5$.

Thus

$$2^{\frac{1}{r}} \text{ or } \sqrt[r]{2}$$

expresses the ratio of any interval formed by two adjacent tones in an equally tempered scale of *r* intervals. Moreover the intervals which any tonic (arbitrarily chosen in the case of the equally tempered scales) forms with the successive ascending tones above it, are, respectively:

$$2^{1/r}, 2^{2/r}, 2^{3/r}, 2^{4/r}, \ldots\ldots\ldots 2^{(r-1)/r}, 2^{r/r} \text{ or } 2.$$

[2] This ratio is physically defined as the interval itself.

From these facts we derive two general formulas: (A) expressing the physical interval or vibration ratio between 2 tones and (B) the vibration frequency of any tone lying above a given tone, N.

(A) $\quad I \text{ (interval)} = 2^{c/r},$

(B) $\quad V \text{ (vib. freq.)} = N \, . \, 2^{c/r},$

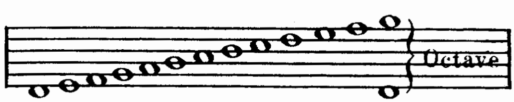
where r = the number of chromatic intervals per octave, in the given system; and c = the number of chromatic intervals separating the two tones whose physical ratio is to be found.

With these formulas we can express the various intervals of any equally tempered scale.

NOTATION OF SCALES.

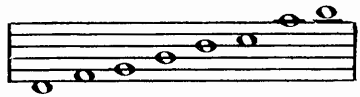
In considering the great number of scales of which we have learned in the previous section we are confronted with the problem of their notation. Our present notation is really suited for seven scales only; namely, the major scale and the scales formed by cyclically rotating the permutation of the intervals of the major scale, that is, the Dorian, Phrygian, Lydian, Mixolydian, etc. We cannot write even a minor scale without the use of an accidental. Then with regard to the 1483 other scales, because of this great number and variety, we cannot do more than make general statements.

We realize, to begin with, that the ideal notation of our present-day music should be one which is designed to eliminate the inconveniences of accidentals. Such a notation would be naturally one designed from the chromatic scale; and because the chromatic scale contains all of the 1490 other scales of the duodecimal system, it would be adaptable, in a perfect sense, to all of these scales. We could accomplish the notation of the chromatic 12-tone scale with a six-line staff giving each degree a separate line or space, as shown:



The major scale on this staff would be:

The minor scale would be:



The mental picture we obtain of the relation of the intervals of these two scales in this manner is alone an advantage. Furthermore, in the six-line staff notation we are less bound to avoid deviations from our chosen scale; we are freer to escape from the tyranny of sharps and flats. An abhorrence of accidentals has always tied us to our chosen scales. Other advantages of this notation could be cited, but the chief one is, of course, that merely through the addition of another line (which does not confuse us optically) we are able entirely to avoid accidentals.

However, the difficulty of introducing this system into common use would be almost too great to be overcome. An attempt at this could be likened to the recent attempts at introducing a universal language; for were we all to learn a universal language we would still have to retain a knowledge of the old for its literature. We are therefore compelled to adjust our new scales to the common notation of the five-line staff.

We may eliminate from consideration not only the major scale and those scales formed by a *cyclic rotation*[3] of its permutation of intervals, but also the minor scale with its corresponding scales formed by a similar cyclic rotation. This suggests to us a process that will greatly simplify our whole problem. We see that the notation for one scale is suitable for all other scales formed by a cyclic rotation of the permutation of its intervals. The number of these scales will depend upon the number of tones or intervals in the original one. The notation for a scale of n tones or n intervals will serve for $(n-1)$ cyclically related scales. Thus one notation serves for n scales.

We realize that out of a certain combination of intervals we may form more than one cyclic group, for some combinations have as many as 168 scales while in no cyclic group can there be more than 11 different scales. A formula with which we may calculate the number of cyclic groups in each combination is:

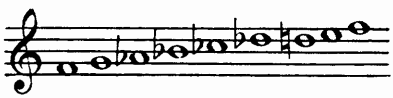
$$G=(n-1)!/n_1!\,n_2!\,n_3!\ldots.$$

where n is the number of intervals in a combination, and n_1, n_2, n_3,

[3] The term defines itself. A *cyclic rotation* of a permutation is one in which the terms are always written in the same order, but each successive permutation begins with the second term of the preceding one. The following is a *cyclic* group of permutations: A B C D, B C D A, C D A B, D A B C.

etc. are the numbers of times respectively which certain intervals are repeated in the combination. There are few exceptions to this formula, all of which are of one type. The erroneous type is that in which $(n_1+n_2+n_3+n_4\ldots)$ exceeds $(n-1)$. These exceptions often cause fractions which cannot be integrally expressed. In cases of this exception we must find our number of cyclic groups by actual trial. But if we have found one signature suitable for each whole cyclic group of scales we have, in general, shown only one-twelfth of possible signatures, for in most cases a different signature is necessary for each chromatic degree. Only in equipartite scales are fewer signatures than twelve necessary to each group. If a scale is bipartite only six signatures are necessary; if tripartite, four; if quadripartite, three; and if sexpartite, two.

As we are considering these 227 scales representing cyclic groups primarily for their notation, we are confronted with the question, what signature shall we give to a work based on a scale like the following?



None of our conventional signatures for major scales will apply to this scale; for we see the three essential signatures are:

d flat being unnecessary as a signature because it is cancelled immediately, the scale being an 8-tone scale, which necessitates the repetition of one note.

We will find that most of the scales, like this one, will require signatures other than those which we have employed for our major and minor modes. Consequently we will not try to reconcile our customary signatures with those natural to the new scales. Therefore, in order to make a signature for any scale on any degree, write down those accidentals which appear in the notation of the scale, omitting those accidentals only which are cancelled as the scale continues. We may rightly call this a *natural* system of signatures.

Concerning the method of finding each scale representative of a cyclic group for a given scale degree, the following means are perhaps the simplest:

1. Choose an interval which occurs singly in the combination and place it in the lowest position in the scale.

2. Permutate the other intervals above it in every possible way.

3. Each permutation, with the first interval remaining in a fixed position, will form a desired scale.

4. When no interval occurs singly in the combination there is no rule which applies generally; but because of the small number of combinations of this character the desired scales can be easily found by trial.

There is little need for investigating the problems of notation of N-tonal systems until such systems come into use. Solutions to such problems are really simple and arbitrary. Suffice it to say, there is no need of retaining the five-line staff for N-tonal notation. It would be unfortunate if one were compelled to read a totally new system of intervals from a staff with which one would constantly associate accustomed intervals.

Although it may seem strange that so much attention is paid a subject like the formation of scales, there is nevertheless justification in an investigation of this sort. A scale has far greater importance than the mere sequence of tones comprising it would imply. Practically all of the hundreds of melodies we know can be formed, almost without accidentals, from the major and minor scales. Virtually all of the common harmonies can be constructed from these modes. The vast amount of musical thought and feeling has until recently expressed itself in major and minor. But the chromatic scale offers a much wider field of expression; for it contains not only the major and minor scales, but over fourteen hundred others. Nevertheless, although the chromatic scale has become the basis for modern harmony, melody does not seem to flow freely chromatically. Our musical speech continually demands some simple group of tones and larger intervals. Without some limitation more binding than the chromatic scale, we are helplessly confused with the wealth of possibilities. Such limitations are found among the multitude of scales derived synthetically from the chromatic.

Debussy and some of his colleagues have made their idiom or "dialect," as it were, the whole-tone scale. This one scale, because of its uncertain "tonality," and its "color," has been the outstanding characteristic of the French impressionists.

A few other scales, such as the Greek "modes," which are all cyclically related to the major scale, have been the basis for numerous works. On the whole, there is no reason why other scales, among the vast number shown to exist, should not become equally important idioms of expression.

Limitations of space unfortunately prevent me from tabulating completely the fourteen hundred and ninety scales of the duodecimal system.

Concerning the N-tone scales, it is well to consider for illustrative purposes the words of Busoni in regard to his tripartite tone scale:[4] "The tripartite tone," says he, "has for some time been demanding admittance, and we have left the call unheeded." With the tripartite tone he encounters a difficulty which will be found also in considering other N-tone scales. He says we would lose through the tripartite tone the minor third and the perfect fifth. Now a chromatic scale in which the most important intervals do not occur (intervals whose ratios are expressed as quotients of the smaller integers) will never form quite as valuable a system as a chromatic scale that contains them. Realizing this, Busoni has attempted to reconcile the 12-tone with the 18-tone system; that is, a system of bipartite tones with one of tripartite tones. His solution is naturally a 36-tone scale involving the sexpartite tone. To entertain any hopes for a system of sexpartite tones seems to me futile. A system of 24-tone chromatics might be better reconciled with our duodecimal system. This example merely shows us that we cannot attempt to reconcile the N-tonal systems with each other or with the duodecimal system. An 18-tone chromatic system is perhaps next in importance to the duodecimal system, but it is comprehensive and important enough in itself, even though it does not contain minor thirds and perfect fifths.

Again we must consider how we are to produce these tones, as Busoni has mentioned in regard to his tripartite tone scale. For experiment and a training of the ear to the tripartite tone Busoni recommends Dr. Thaddeus Cahill's dynamophone, an instrument which would, however, be very difficult to obtain or to construct. A Seebeck's siren with a special disk for each system would be a good substitute. The number of holes in each circular row could be mathematically computed with the help of the formulas:

$$I = 2^{d/r} \text{ and}$$
$$V = N \, . \, 2^{d/r},$$

[4] For Busoni's statements read his *Sketch of a New Esthetic of Music*, New York, 1911.

which are explained in previous pages. A motor to revolve the disk would furnish a constant speed of rotation.

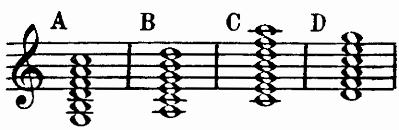
Such experiments would furnish means of acquiring a sense of intervals other than those to which we are accustomed; but, in Busoni's words, "only a long and careful series of experiments and a continued training of the ear can render this material approachable and plastic for the coming generation, and for art."

A NEW HARMONY.

Our present system of harmony, the system of chords (harmonies formed of superimposed thirds), is deficient in two important respects. First, it is often unwieldy; and second, it is not fully comprehensive. This latter shortcoming is partly responsible for the former, since it is true that we may represent certain harmonies, seemingly not within the scope of our system, in complex ways. To illustrate: let us examine the various unsatisfactory ways of describing the simple harmony:

If the harmony is a chord, we must be able to build it up by superimposing thirds. But no complete chord exists that contains each of these and only four notes.

But there are chords containing more than four notes which contain the notes of the harmony, such as the following:



From these chords we may strike out those notes foreign to the harmony and derive what we call an *incomplete* chord. The harmony:

may therefore be termed an *incomplete 11th chord* (as in A, B, or D) or an *incomplete 13th chord* (as in C). If we are willing to recognize an incompleteness of this sort as mathematically rigid, we must still admit that such a naming of the harmony as an incomplete 11th does nothing more than justify its existence among chords. It does not name the harmony for there are innumerable incomplete 11ths. To name the 11th chord in each case is difficult, the general method being that of determining upon what degree of the major or minor scale it is built. But, again, must we consider all harmonies in the light of the major or minor scales to-day when many other scales are being used? Furthermore we must find where the incompleteness lies. Lastly one would suppose that every harmony has one fundamental position, but here are four. One should be able to tell what sort of inversion of the fundamental harmony the one in question is. How is this possible when the harmony in its position is a different inversion with each fundamental?

If we allow ourselves the latitude of recognizing diminished thirds, we may say the harmony is composed of two diminished thirds separated by a minor third. Taking this liberty we might have a *specialized chord* or so-called *altered chord,* but how shall we describe any particular one? Moreover, it is false to assume that diminished thirds are thirds at all; for they are seconds.

Sometimes, if the harmony is preceded or followed by others, we may analyze it under our present system by considering certain tones as "passing tones," "suspensions," "afterbeats," "syncops," "organ-point," etc. The awkward system of figured bases sometimes affords a means for expressing simpler chords.

If such is the fate that a simple harmony like the above suffers in analysis, what lot befalls the multitiude of more complex harmonies? The best that modern analysis can do for them is to treat them in relation to surrounding harmonies. Even then, "unresolved suspensions," etc. are continually met with in modern music. If harmony is "that which sounds together," we should be able to define any combination of simultaneously sounding tones, whether this combination is surrounded by others or not. A note suspended from a consonant to a dissonant chord is sounding in the second as well as in the first harmony. Does not an organ-point form a separate harmony with each of a series of chords "moving through it," even though these chords are dissonant with the organ-point? A harmony is a harmony whether dissonant or consonant. Yet of the vast majority of dissonant harmonies few can be adequately named and classified in themselves.

The chord system is adequate in analysis of older works only. It can give only a superficial analysis of modern works.

A more important objection even than that of inadequate nomenclature is that by reason of our use of the chord system we are hindered in enlarging our scope of harmonies. The conception of harmonies given us by this system restrains us from enlarging our harmonic vocabulary. Bred in the chord system, we are prone to regard any harmony which is not chordic in construction as a mere variance of some "simple" chordic form. Many a stereotyped theorist would shudder at the notion of giving the above harmony a prolonged and separate existence. It must be immediately resolved into a stable form; the tonic triad of G major, etc., etc. Are we blind to the existence of harmonies not made up of superimposed thirds? Shall we refuse to recognize non-chordic harmonies merely for the technical reason that we employ a system of superimposed thirds, which was an expedient solution to theoretical problems two centuries ago? Because of the limitations of our present system, a vast number of harmonies remain to-day virtually undiscovered. Although many have been employed passingly and subconsciously, few have been employed deliberately, few are spoken of as a part of the composer's vocabulary.

Fully realizing the importance of the chord system in the analysis of older works (for these were conceived in the spirit of the chord system), I believe it is important that a new and fully comprehensive system should supplement it, a system that would prove adequate for the analysis of modern writings. Whereas the old system embraces chords only, the new should embrace all harmonies. The chord scheme would then take its place as a sub-system of the more general and all-inclusive system.

Just as the present method is more than a mere scheme of nomenclature, so the one which I propose should be considered as affecting more than the mere naming of harmonies. The chord system teaches us that all harmonies are chords, are built by imposing major and minor thirds upon each other. The proposed system should, as will be shown, teach us to recognize harmonies which are built by superimposing any intervals. It should teach us a broader conception of harmonies and, as I believe, a more valuable one, since the importance of a notion usually depends upon its generality.

To-day the modern composer habitually employs the twelve-tone scale as the source of his harmonic invention. The abundance of accidentals in our modern composition is superficial but none the less accurate evidence of the passing of the feeling for the diatonic modes. To-day there are also a few scales which are formed of new arrangements in the intervals of our duodecimal system.

But the octave still remains the common basis for all scales now used; each scale repeats itself at successive octaves.

It seems only natural that we make this interval which is of the greatest importance because it has the simplest ratio, the basis for our harmony. We may therefore call its interval unity.[5]

Having established the octave as the basic interval, and having assigned to it the number one, we turn our attention to the lesser intervals. The semitone, since it is one-twelfth the gradation of the octave, will be known as the interval, one-twelfth. The "whole tone" becomes two-twelfths. Tabulating all of our intervals in their old and new nomenclature we have:

DIATONIC NAME	NATURAL OR CHROMATIC NAME
Minor Second	One Twelfth
Major Second	Two Twelfths
Minor Third	Three "
Major Third	Four "
Perfect Fourth	Five "
Augmented Fourth	Six "
Perfect Fifth	Seven "
Minor Sixth	Eight "
Major Sixth	Nine "
Minor Seventh	Ten "
Major Seventh	Eleven "
Octave	One.

Although many of these fractions expressing intervals are not reduced to their simplest form, it is of advantage to retain the common denominator, twelve; for if all intervals can be expressed as quotients of variable integers and the constant twelve, we need consider only the numerators and eliminate the common denominator. Thus the intervals, one-twelfth, two-twelfths, three-twelfths etc., may be called respectively, one, two, three, etc. It is clear that this nomenclature is founded entirely upon the chromatic scale since every interval is measured as a multiple of the intervals comprising the chromatic scale.

In naming harmonies having more than one interval, the advantages of the chromatic nomenclature are immediately apparent. For instance, the major triad is said to be formed of a minor third placed above a major third. In other words the interval 3 is placed above the interval 4. Thus the major triad in fundamental position

[5] For simplicity we call the interval unity, although the physical interval of the octave is 2.

is a $^{3}{}_{4}$ or 4-3 harmony. Likewise, the minor triad is a 3-4 harmony; the dominant sept chord in fundamental position, a 4-3-3 harmony; and the dominant sept chord in its first inversion is a 3-3-2 harmony. The harmony under previous discussion is, in the form

a 2-3-2 harmony.

This change in nomenclature means more than is at first apparent. It means an expansion of our conception of harmonies which may, perhaps, offset the limitations felt in the minds of many who can think of music only as varying arrangements of groups of superimposed thirds. We may freely think of any harmonies as being composed of superimposed intervals of any sort, instead of being shackled by considering every harmony made up of superimposed thirds, or inversions of these. Our vocabulary of harmonies, instead of embracing only chords, will embrace all harmonies.

It may be well here to forestall a possible objection that the chord system is the nearest to the ideal from the physical point of view. It is true that chords are physically the "cleanest" harmonies, i. e., their tones have the simplest vibration ratios. It may be said from this that the system of building up thirds is founded not upon an arbitrary choice, but upon an acoustic basis. But I answer that from this point of view it is immaterial whether we think of major and minor thirds or of 4s and 3s in building up the most important harmonies. If thirds and sums of thirds are the most important intervals, then, after we have learned the chromatic nomenclature, 3s, 4s, 6s and 7s will come to be recognized as the most important intervals. There is no ground for any charge that the chromatic nomenclature is more empiric and less scientific than the chord system.

Again, is not the chromatic nomenclature a simple and an accurate method for naming and classifying any harmony? Instead of grappling with such harmonies as these:

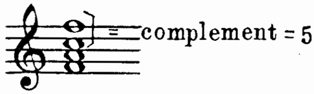
considering them in relation to surrounding harmonies, and in themselves by devious ways, we simply name them chromatically as consisting of superimposed intervals. They are respectively: 6-4-6-4 and 5-5-4-4 harmonies.

Our next task is to study more closely the nature of harmonies and to discover suitable means for systematically finding all of these harmonies.

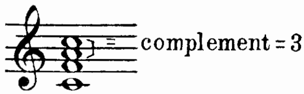
First we will recognize a tone essentially the same whether its vibration frequency is increased or diminished by a power of two; for the ordinary ears hear it as essentially the same. For this reason, we are compelled to regard an inversion of a harmony only as a different position of the harmony and not as a different harmony. In the chord system we decide upon one position of a harmony which we call close fundamenal, namely that position in which the harmony's root is in the base. (There is, however, no distinction made between "open" and "close" fundamental positions.) Likewise, and for purpose of classification, we should decide upon a fundamental position of a harmony in our chromatic system of nomenclature. The choice of a fundamental position, though arbitrary, will be made later in the discussion.

The relations between the tones comprising a harmony are intervals. A harmony may be thought of as a combination of intervals. However, a combination of intervals may form more than one harmony. The *major triad* is a 4-3 harmony, that is it is made up of the intervals 4 and 3; yet if we merely reverse the order of these intervals we have another harmony, the *minor triad*. It is clear then that a harmony is one permutation only of a given combination of intervals.

Now it might be thought that one could form all harmonies existing in our duodecimal system by permutating all possible combinations of intervals in all possible ways. However, by doing this we would calculate an immense number of harmonies, very many of which would only be inversions of each other. To avoid the repetitions occasioned by inversions, in classifying all harmonies, we come to the consideration of an extra interval with each harmony. If we invert the major triad or the 4-3 harmony we have first a 3-5 harmony (commonly called the first inversion) and then a 5-4 harmony (called the second inversion). The extra interval to be considered in this case is the interval 5, which is bounded by the highest tone of the triad and the note an octave above the triad's lowest tone, the triad standing in fundamental position—an interval which will be defined as the *complement*. If the triad is in 4-3 position of f, the complement is the interval c-f:



If the triad is in 5-3 position on c, the complement becomes the interval a-c or 3.



The complement will be denoted by the letter R in the following paragraphs. The term will be employed in the discussion of harmonies having any number of tones or intervals.

Now if A represents the lowest interval of a harmony in close position, that is, such a position in which all tones are within the compass of an octave; B the interval above A; C the interval above B, etc., and R the complement, a harmony could be represented thus:

R
.
.
.
. or ABC....R
C
B
A

The first inversion of this harmony would be represented thus:

A
R
.
. or BC..RA
.
C
B

the second inversion thus:

B
A
R
.
. or C...RAB[6]
.
.
C

[6] Each different letter does not necessarily denote a different sized interval.

We can think of all the different positions of the same harmony as arranged clockwise in a circle. Taking the intervals in clockwise order we find a different inversion taking each letter as a starting-point. It is clear that the inversions of a harmony are cyclic rotations of the permutation of intervals forming the fundamental position, whichever it may be. Therefore if we desire to form all possible harmonies from a given combination of intervals, we employ only those permutations of the given combination which are not cyclically related; that is, out of each group of permutations which are cyclic rotations of each other we select only one permutation as representative of a single harmony.

The Number of Harmonies with a Given Combination.

Assume we are given a harmony A-B, in which A and B are of different magnitude. Furthermore let us make A and B such intervals that the complement (R) of A-B, is unequal in magnitude to either A or B. Including R in our letter representation, we denote the harmony as A-B-R. The minor (3-4-5) and major (4-3-5) triads are good examples of such a harmony. We have already seen that the inversions of a harmony are cyclic rotations of its arrangement of intervals. Conversely, if two or more harmonies are cyclically related they are only different positions or inversions of one and the same harmony.

Since we are now engaged in finding how many harmonies can be formed from a given combination of intervals, our problem becomes one of finding the number of cyclically unrelated permutations of the given combination.

Let us experiment with our harmony A-B-R. The permutations of the combination are:

A B R	B R A	R A B
A R B	B A R	R B A

The arrangements in the upper row are cyclically related; likewise those in the lower; yet no one of the permutations in the upper row is related cyclically to any one in the lower row. Hence, only two harmonies: A-B-R and A-R-B, can be formed of the combination A, B, and R. We observe that both harmonies are represented in each vertical group; moreover in each vertical group one letter occupies the same position (i. e., first in this case) in both harmonies while the other two letters are permutated.

Let us experiment in like manner with a combination of 4 intervals (R included as one), in which all the intervals are of different magnitudes. We represent its parts as A, B, C, and R. Its permutations are:

A B C R	B C R A	C R A B	R A B C
A B R C	B R C A	C A B R	R C A B
A C B R	B R A C	C B R A	R A C B
A C R B	B A C R	C R B A	R B A C
A R B C	B C A R	C A R B	R B C A
A R C B	B A R C	C B A R	R C B A

Here, too, the permutations have been so arranged that those in any one horizontal row are cyclically related, while no permutation in one horizontal row is cyclically related to any one in any other horizontal row. Thus any one of the vertical columns contains all possible harmonies: in this case 3! or 6. It will be noticed here as before, that by retaining one letter in a stationary position throughout while permutating the remaining letters, all possible harmonies are formed from the given combination; for, although many of the permutations of the remaining letters are cyclically related, the stationary letter will bear a different relation to the other letters in each case. In each vertical column one letter is held in the same position, allowing the remaining 3 letters to be permutated in 3! different ways.

With a combination of two intervals (R included) we form 1! or 1 harmony; with three intervals (R included) we form 2! or 2, with four intervals we form 3! or 6, etc. Thus with n different intervals we form $(n-1)!$ different harmonies.

But this formula does not apply to combinations in which two or more intervals are alike; and such combinations are by far more numerous than the others. In fact, no harmony can have as many as five or more different intervals, provided of course that all its tones are bounded by the octave. To illustrate, we find the sum of the five smallest intervals (1, 2, 3, 4, 5) to be 15, which exceeds the octave 12 by 3.

To experiment with harmonies in which two or more intervals are alike let us take the combination: A, B, B, and R. Its permutations are:

A B B R	B B R A	B R A B	R A B B
A B R B	B R B A	B A B R	R B A B
A R B B	B A R B	B B A R	R B B A

Here again the horizontal rows contain cyclically related permutations while the vertical columns do not. We find all three of the possible harmonies represented in any one vertical group (in each of which one letter has a stationary position throughout). However, since the combination contains two B's there are two "B rows," each of which contains the three harmonies. Thus if one B is held stationary while the remaining intervals are permutated, we obtain six instead of three harmonies; showing that our rule of stationaries[7] holds only for singly occurring intervals. But if we hold A stationary, we permutate B-B-R in 3!/2! different ways forming 3 different harmonies.

With the combination A, B, B, R and R we can form the following harmonies: ABBRR, ABRBR, ABRRB, ARBRB, ARBBR, ARRBB, by holding A in the same position and permutating the remaining letters in 4!/2!2! (=6) different ways.

The general formula then for the number of harmonies to a given combination in which at least one interval occurs singly is:

$$\text{H (harmonies)} = (n-1)!/n_1!\,n_2!\,n_3!\ldots\ldots$$

where n is the total number of intervals (R included) in the combination, and n_1, n_2, n_3, etc., are the respective number of times which certain intervals are found. This is the general formula for the number of harmonies to a combination, as the large majority of combinations contain singly occurring intervals. To find the number of harmonies in a combination containing no singly occurring intervals actual trial must be resorted to; any formula for this would be beyond the scope of this work. As an example of the error to which the formula leads if it is applied to combinations having no singly occurring interval, we will apply it to the combination 3 A's and 3 B's. Here

$$(n-1)!/n_1!\,n_2!\ldots = 5!/3!\,3! = 1.2.3.4.5/(1.2.3)(1.2.3) = 10/3$$

The result is an impossibility, since a fractional number of harmonies cannot exist.[8]

The Number of Possible Combinations.

We will henceforth regard a harmony as in a *prime* position if its tones are reduced to within the compass of an octave.

[7] That is, the method of holding one interval stationary and permutating the remaining intervals in all possible ways to obtain the several different harmonies.

[8] We remember that throughout the last pages we have considered the complement R as one of the n intervals of a combination. It might be supposed that we could now eliminate R from consideration and thereby make our formula, $n!/n_1!\,n_2!\ldots$ This could not be done generally, since R might be an interval having a magnitude equal to that of another interval (i. e., A, B, or C, etc.) in the combination; in that case it would necessarily figure in the denominator of the fraction.

No harmonies in prime position can consist of less than two or more than twelve tones, in our duodecimal system. Hence no prime harmony can have less than two (R included) or more than twelve intervals. Now the number of combinations possible within the octave could be computed, but the result would be of no value, since in finding the number of harmonies we must treat each combination separately. Furthermore, mere numbers interest us only speculatively while a concrete method of obtaining all harmonies is of real value.

To simplify the task of finding the combinations possible with the twelve units within the octave, we will treat separately those groups of combinations having a different number of tones or intervals. A separate table will be made for each group of combinations having a certain fixed number of intervals. It is, of course, evident that the sum of the intervals of each combination equals the octave 12, since R is always one of the combination's intervals. In the tables the vertical columns contain the respective intervals, varying in magnitude as the number of intervals of the combination permit. In the horizontal rows are found the combinations. If a number greater than 1 is found in a combination, it indicates how many times the interval in whose vertical column it lies is repeated in the combination. Thus a number 3 lying in a column headed by the number 2, indicates that the combination contains the interval 2 (or the whole tone) thrice. The tables follow.[9]

TABLE I; N (NUMBER OF INTERVALS INCLUDING R)=2

	Various Intervals											Calculations of Harmonies	(No. of Harm.) H
	1	2	3	4	5	6	7	8	9	10	11		
1	1										1	$H=(n-1)!=(2-1)!$	1
2		1								1		“	1
3			1						1			“	1
4				1				1				“	1
5					1		1					“	1
6						2						“	1
												Total	6

[9] The combination numbers found to the left of the respective combinations are only for future reference.

TABLE III; N=4

	1	2	3	4	5	6	7	8	9	CALCULATIONS OF HARMONIES	H
1	3								1	H=3!/3!	1
2	2	1						1		H=3!/2!	3
3	2		1				1			"	3
4	2			1		1				"	3
5	2				2					By trial	2
6	1	2					1			3!/2!	3
7	1	1	1			1				3!	6
8	1	1		1	1					"	6
9	1		2		1					3!/2!	3
10	1		1	2						"	3
11		3				1				3!/3!	1
12		2	1		1					3!/2!	3
13		2		2						By trial	2
14		1	2	1						3!/2!	3
15			4							By trial	1
										Total	43

TABLE II; N=3

	1	2	3	4	5	6	7	8	9	10	CALCULATIONS OF HARMONIES	H
1	2									1	H=(3—1)!/2!	1
2	1	1							1		(3—1)!	2
3	1		1					1			"	2
4	1			1			1				"	2
5	1				1	1					"	2
6		2						1			2!/2!	1
7		1	1				1				2!	2
8		1		1		1					"	2
9		1			2						2!/2!	1
10			2			1					"	1
11			1	1	1						2!	2
12				3							By trial	1
											Total	19

TABLE IV; N=5

	1	2	3	4	5	6	7	8	CALCULATIONS OF HARMONIES	H
1	4							1	H=4!/4!	1
2	3	1					1		4!/3!	4
3	3		1			1			"	4
4	3			1	1				"	4
5	2	2				1			4!/2! 2!	6
6	2	1	1		1				4!/2!	12
7	2	1		2					4!/2! 2!	6
8	2		2	1					"	6
9	1	3			1				4!/3!	4
10	1	2	1	1					4!/2!	12
11	1	1	3						4!/3!	4
12		4		1					4!/4!	1
13		3	2						By trial	2
									Total	66

TABLE V; N=6

	1	2	3	4	5	6	7	CALCULATIONS OF HARMONIES	H
1	5						1	H=5!/5!	1
2	4	1				1		5!/4!	5
3	4		1		1			"	5
4	4			2				By trial	3
5	3	2			1			5!/2! 3!	10
6	3	1	1	1				5!/3!	20
7	3		3					By trial	4
8	2	3		1				5!/2! 3!	10
9	2	2	2					By trial	16
10	1	4	1					5!/4!	5
11		6						By trial	1
								Total	80

TABLE VI; N=7

	1	2	3	4	5	6	CALCULATIONS OF HARMONIES	H
1	6					1	H=6!/6!	1
2	5	1			1		6!/5!	6
3	5		1	1			"	6
4	4	2		1			6!/2! 4!	15
5	4	1	2				"	15
6	3	3	1				6!/3! 3!	20
7	2	5					By trial	3
							Total	66

TABLE VII; N=8

	1	2	3	4	5	CALCULATIONS OF HARMONIES	H
1	7				1	H=7!/7!	1
2	6	1		1		7!/6!	7
3	6		2			By trial	4
4	5	2	1			7!/2! 5!	21
5	4	4				By trial	10
						Total	43

TABLE VIII; N=9

	1	2	3	4	CALCULATIONS OF HARMONIES	H
1	8			1	H=8!/8!	1
2	7	1	1		8!/7!	8
3	6	3			By trial	10
					Total	19

TABLE IX; N=10

	1	2	3	CALCULATIONS OF HARMONIES	H
1	9		1	H=9!/9!	1
2	8	2		By trial	5
				Total	6

TABLE X; N=11

	1	2	Calculations of Harmonies	H
1	10	1	H=10!/10!	1

TABLE XI; N=12

	1	Calculations of Harmonies	H
1	12	By trial	1

Let us tabulate the number of harmonies found in each respective table:

Table No. 1	Intervals 2	Total No. H 6
" " 2	" 3	" " " 19
" " 3	" 4	" " " 43
" " 4	" 5	" " " 66
" " 5	" 6	" " " 80
" " 6	" 7	" " " 66
" " 7	" 8	" " " 43
" " 8	" 9	" " " 19
" " 9	" 10	" " " 6
" " 10	" 11	" " " 1
" " 11	" 12	" " " 1
		Total 350

We notice that as the number of intervals increases to 6 the number of harmonies increases, while as the number of intervals increases beyond 6 the number of harmonies decreases. Thus, six-tone harmonies (or harmonies of six intervals including R) are most numerous; five- and seven-tone harmonies next in number; four- and eight-tone harmonies next; etc. More harmonies can be formed from combination 4 Table VII than from any other combination. From it we obtain 21 harmonies. Finally we see that the total number of harmonies in the duodecimal system is 350.

The harmonies of least dissonance will be those having the fewest small intervals. There are 9 harmonies having intervals no smaller than 3 (minor third); there are 28 having intervals no smaller than 2 (major second); and there are 55 harmonies having only one interval, 1 (minor second).

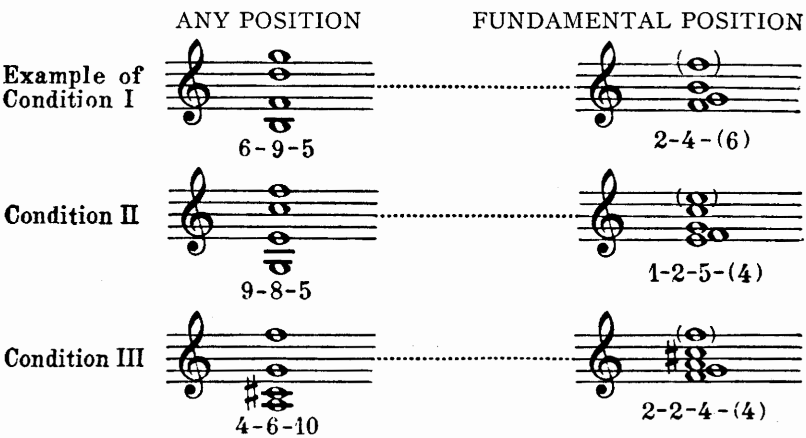
So far we have only shown the number of harmonies with each separate combination. Now it remains to show every harmony on the staff. Means have been shown for finding the number of harmonies from each combination. We merely retain a singly occurring interval in one position (preferably the lowest) and permutate the remaining intervals. But in notating harmonies we should at least represent them in some standard form; and thus we arrive at the long-delayed decision about fundamental position.

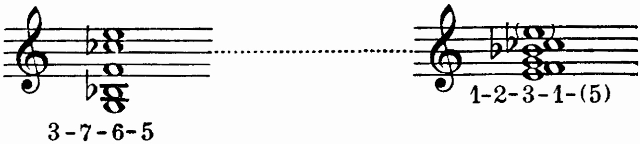
Fundamental Position.

Among the 350 complex harmonies which we have found, there are many—nay, a large majority—which could not be represented as plain chords. Furthermore, since our vocabulary of intervals and harmonies has become a chromatic one, we will no longer attempt to reconcile the limited number of harmonies known as chords, with all of 350 harmonies; hence no attempt to make a chordic position the fundamental form. Arbitrarily we might choose as our fundamental a form in which the *span,* or the interval between the extreme tones of a harmony, is smallest. Or, since we have found it convenient to place any singly occurring interval in the lowest position in forming harmonies from combinations, we might call such a position fundamental. The question is difficult, and although my solution is only arbitrary I believe the fundamental position should be one which satisfies the following conditions:

I. The harmony is prime.
II. The smallest interval (it may be R) occupies the lowest position, and thereby becomes A.
III. In case there exist two or more smallest intervals, the one or more other smallest intervals are nearest A.
IV. In case the two or more smallest intervals are spaced regularly apart, the next smallest interval is nearest A.

Illustrations follow in order respective to the conditions of the definition.

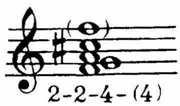


Condition IV 

The first example illustrates how a harmony in more or less spread-out position (left-hand column) is reduced to prime position.

The second illustrates how another harmony (9-8-5) is reduced to prime position, following which it is placed so that the smallest interval (1) occupies the lowest position.

The third illustrates a harmony which, in any prime position, contains two smallest intervals. We are not satisfied with making either of these A; the equivalent of A must be nearest A. Thus in this position:



A's equivalent is nearer A than in the position:

The fourth illustrates a harmony containing two smallest intervals (=1) which are equally separated in any prime position of the harmony. Thus the two intervals (1) are always separated by the interval 5. But because the next smallest interval

 lies nearer  than 

we consider the interval e-f as A.

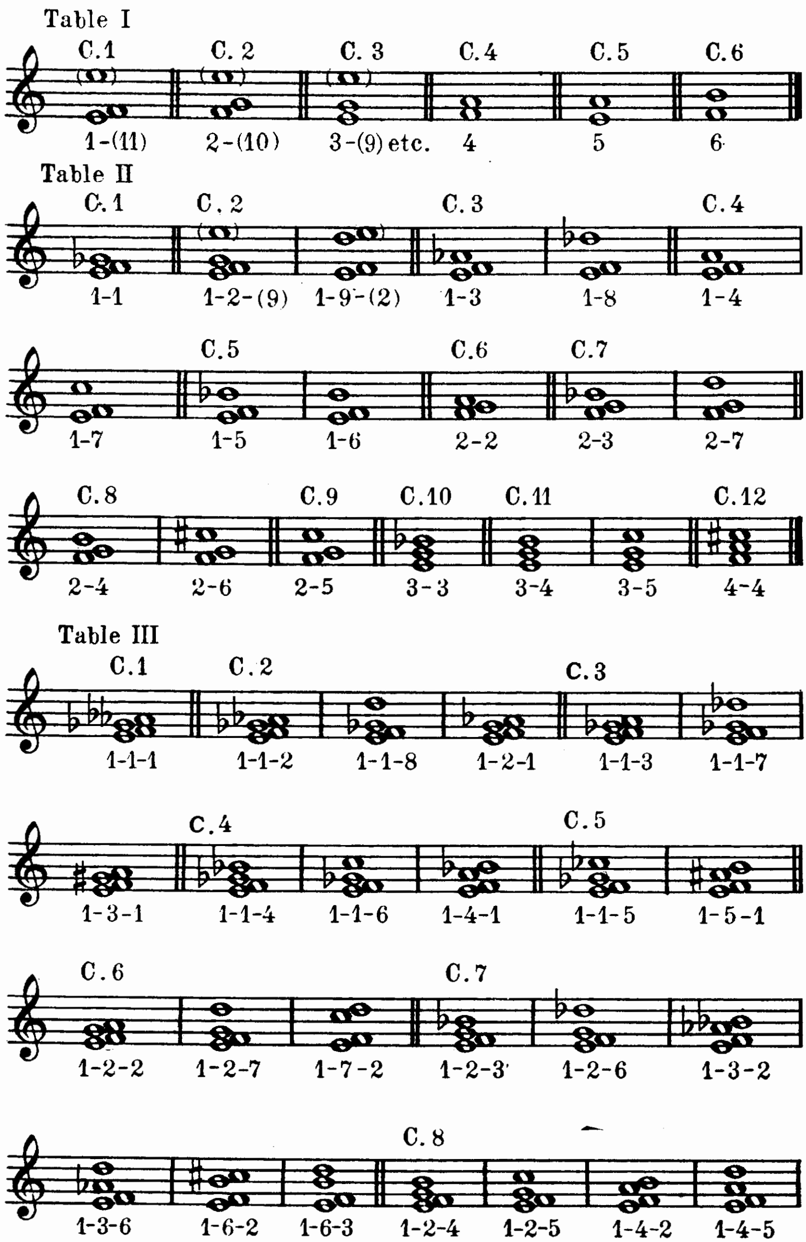
Thus in classifying any harmony, only three short steps are necessary. First we reduce its tones to within the compass of the octave; second, we select from the prime positions the fundamental position; third, we name the harmony according to the chromatic nomenclature.

Having determined the fundamental position, we are prepared to write out on the staff all existing harmonies with the help of the previous tables. Every harmony will appear in fundamental form; while each will be respectively named. The harmonies follow:

ALL EXISTING HARMONIES OF THE DUODECIMAL SYSTEM.

Listed in Fundamental Positions and by Tables.

(The names of the harmonies are written respectively below; the number of combination in which each is found appears above the staff.)



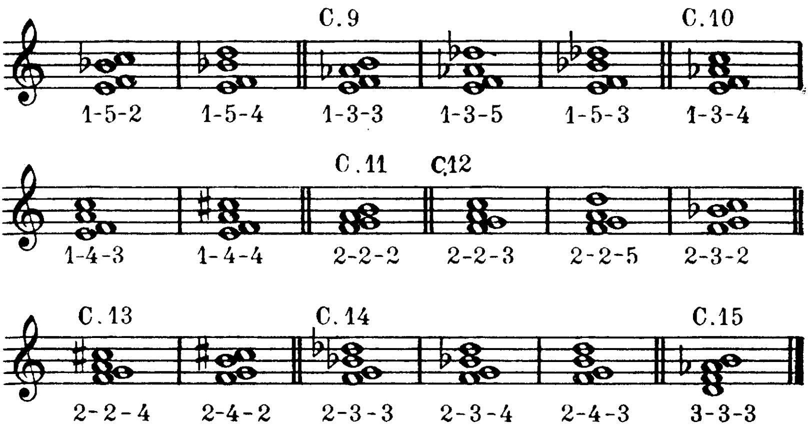
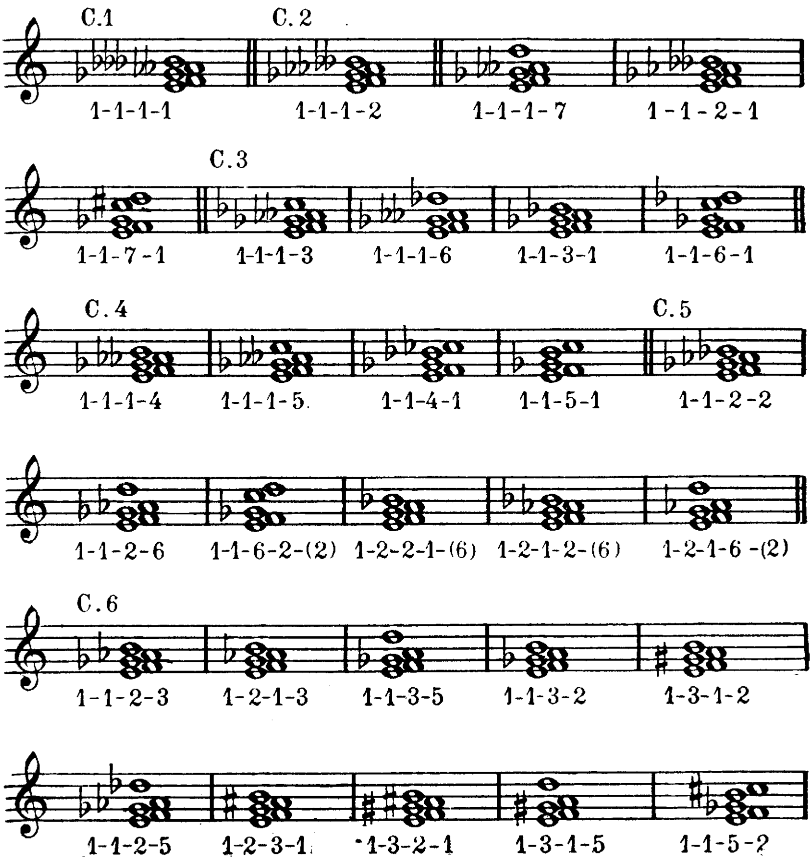
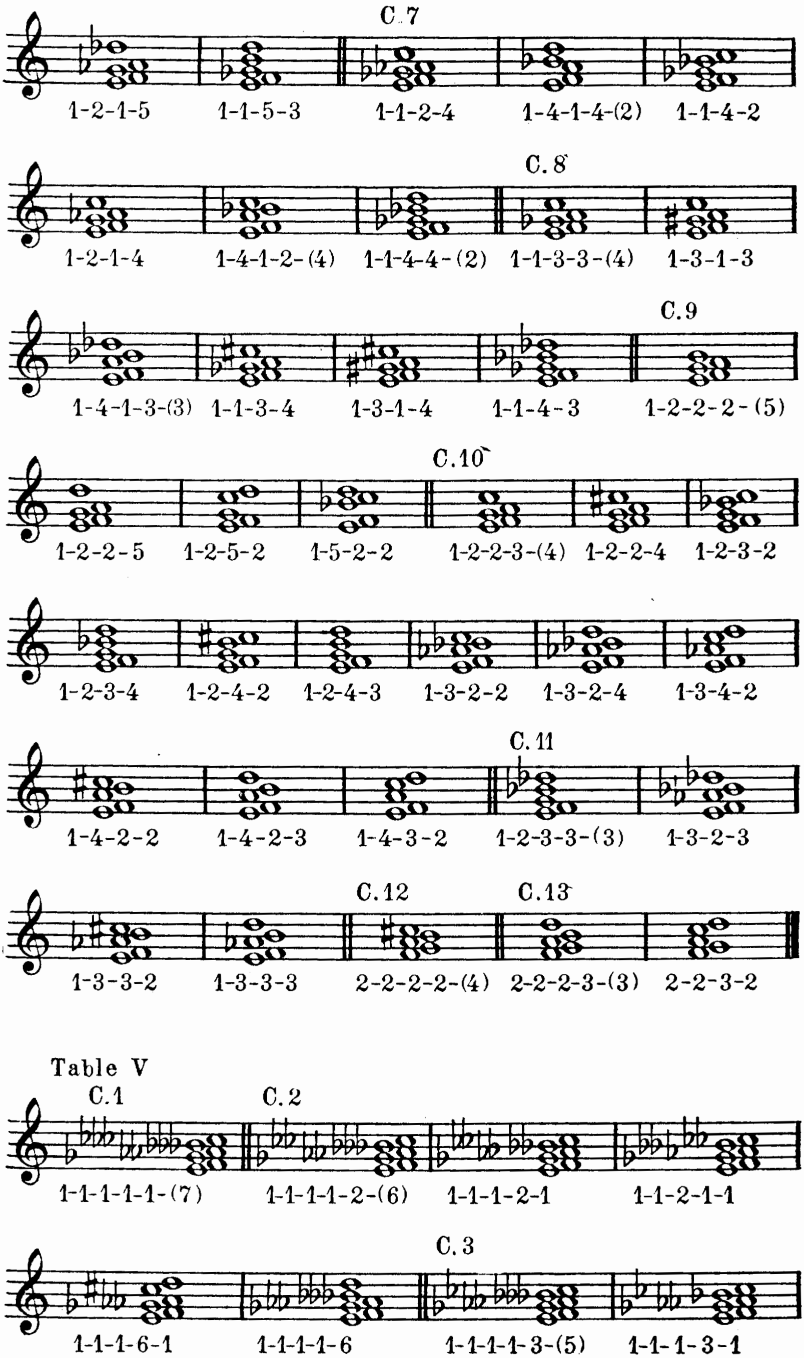
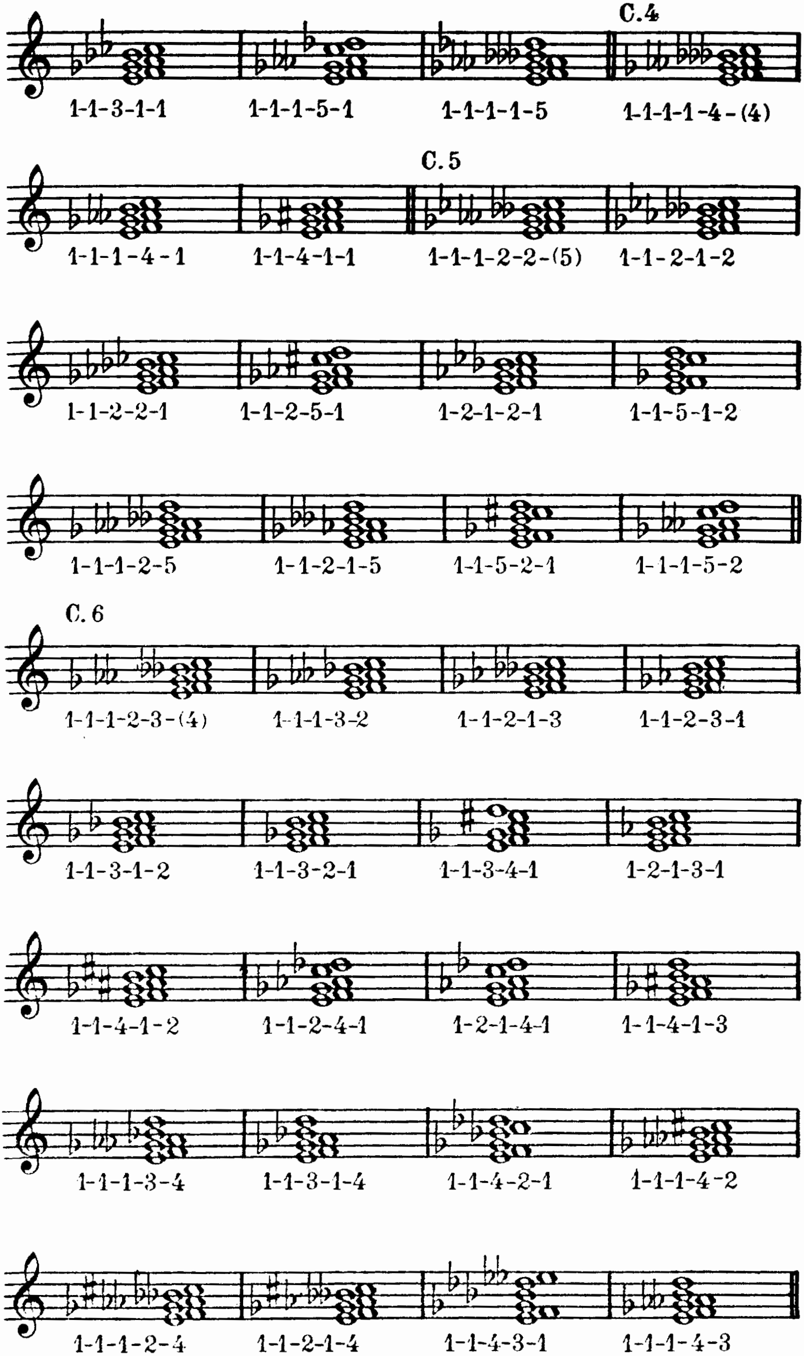
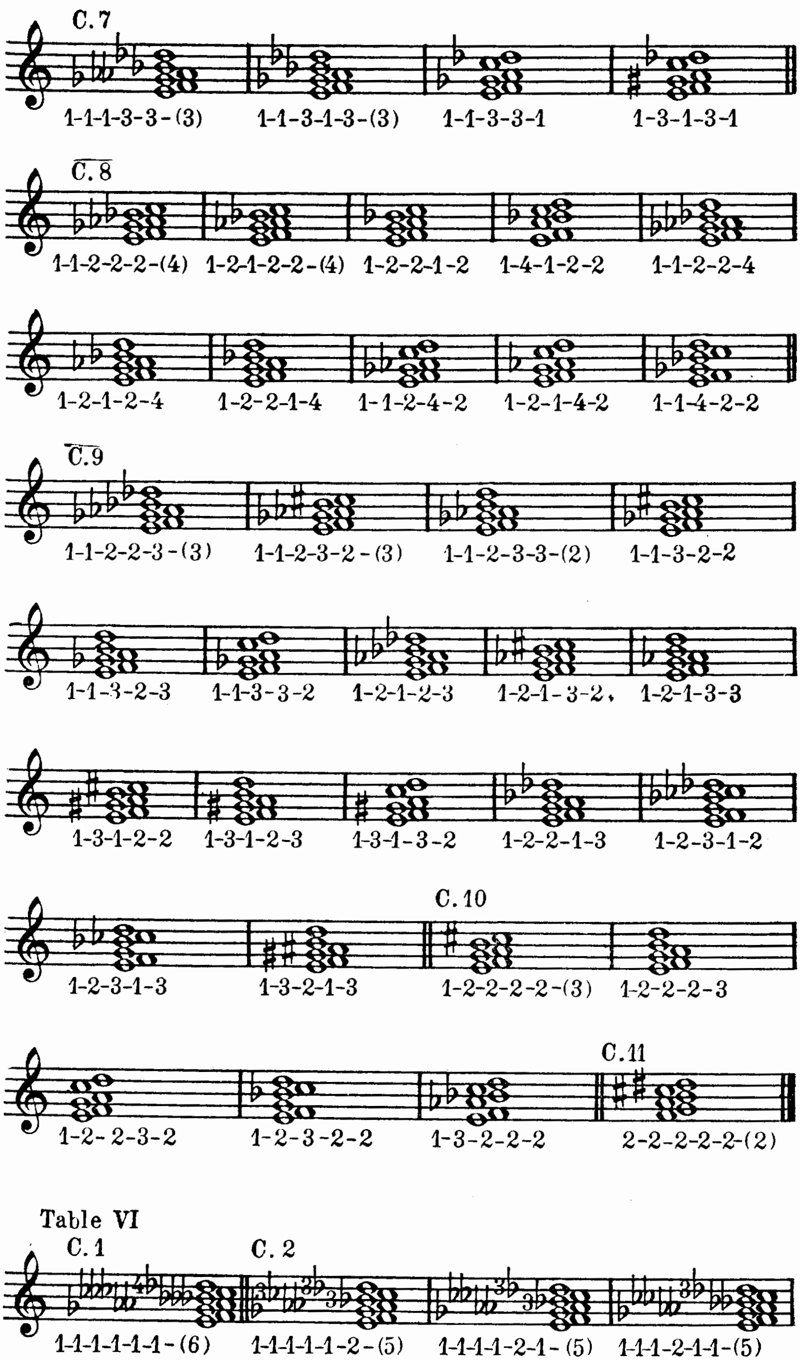

C.9
C.10
1-5-2
1-5-4
1-3-3
1-3-5
1-5-3
1-3-4
C.11
C.12
1-4-3
1-4-4
2-2-2
2-2-3
2-2-5
2-3-2
C.13
C.14
C.15
2-2-4
2-4-2
2-3-3
2-3-4
2-4-3
3-3-3

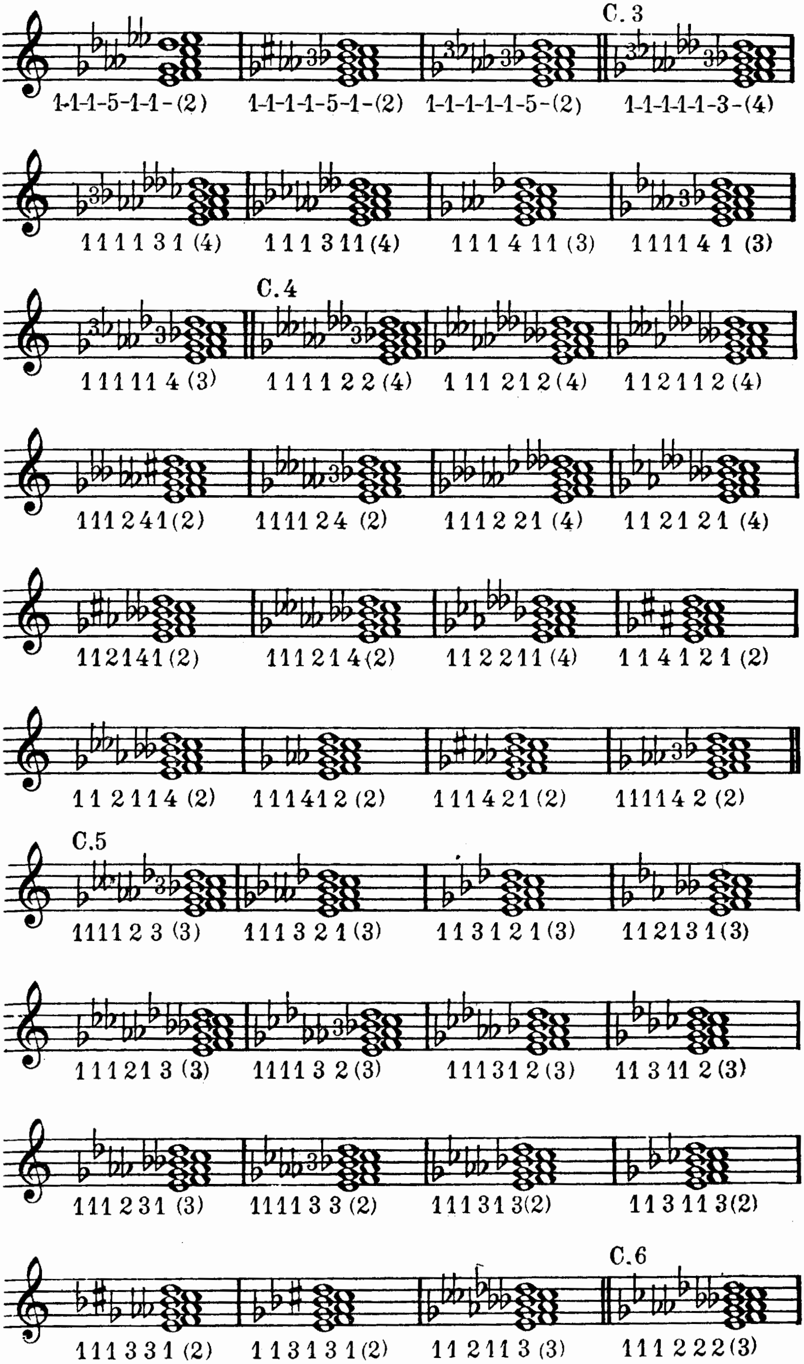
Table IV

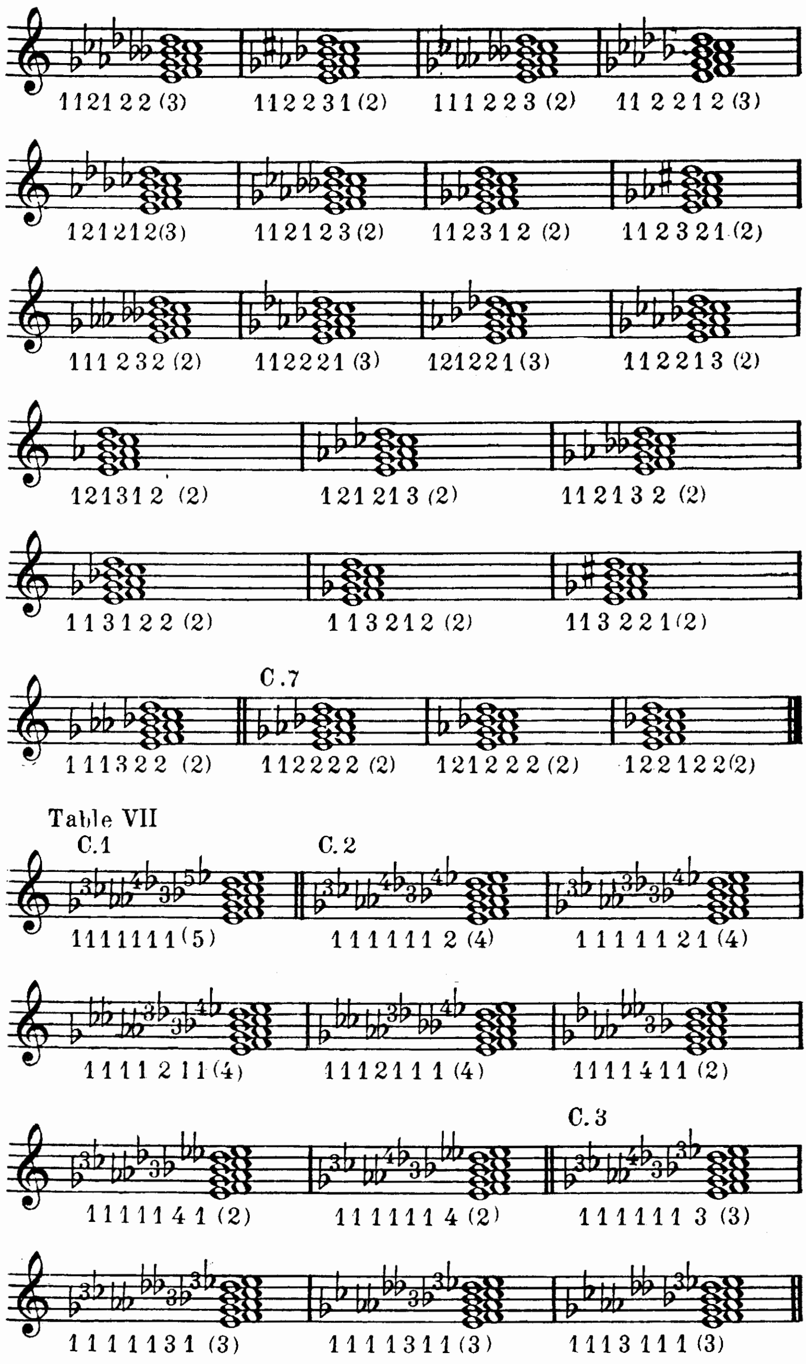

C.1
C.2
1-1-1-1
1-1-1-2
1-1-1-7
1-1-2-1
C.3
1-1-7-1
1-1-1-3
1-1-1-6
1-1-3-1
1-1-6-1
C.4
C.5
1-1-1-4
1-1-1-5
1-1-4-1
1-1-5-1
1-1-2-2
1-1-2-6
1-1-6-2-(2)
1-2-2-1-(6)
1-2-1-2-(6)
1-2-1-6-(2)
C.6
1-1-2-3
1-2-1-3
1-1-3-5
1-1-3-2
1-3-1-2
1-1-2-5
1-2-3-1
1-3-2-1
1-3-1-5
1-1-5-2


C.7
1-2-1-5
1-1-5-3
1-1-2-4
1-4-1-4-(2)
1-1-4-2
C.8
1-2-1-4
1-4-1-2-(4)
1-1-4-4-(2)
1-1-3-3-(4)
1-3-1-3
C.9
1-4-1-3-(3)
1-1-3-4
1-3-1-4
1-1-4-3
1-2-2-2-(5)
C.10
1-2-2-5
1-2-5-2
1-5-2-2
1-2-2-3-(4)
1-2-2-4
1-2-3-2
1-2-3-4
1-2-4-2
1-2-4-3
1-3-2-2
1-3-2-4
1-3-4-2
C.11
1-4-2-2
1-4-2-3
1-4-3-2
1-2-3-3-(3)
1-3-2-3
C.12
C.13
1-3-3-2
1-3-3-3
2-2-2-2-(4)
2-2-2-3-(3)
2-2-3-2
Table V
C.1
C.2
1-1-1-1-1-(7)
1-1-1-1-2-(6)
1-1-1-2-1
1-1-2-1-1
C.3
1-1-1-6-1
1-1-1-1-6
1-1-1-1-3-(5)
1-1-1-3-1


C.4
1-1-3-1-1
1-1-1-5-1
1-1-1-1-5
1-1-1-1-4-(4)
C.5
1-1-1-4-1
1-1-4-1-1
1-1-1-2-2-(5)
1-1-2-1-2
1-1-2-2-1
1-1-2-5-1
1-2-1-2-1
1-1-5-1-2
1-1-1-2-5
1-1-2-1-5
1-1-5-2-1
1-1-1-5-2
C.6
1-1-1-2-3-(4)
1-1-1-3-2
1-1-2-1-3
1-1-2-3-1
1-1-3-1-2
1-1-3-2-1
1-1-3-4-1
1-2-1-3-1
1-1-4-1-2
1-1-2-4-1
1-2-1-4-1
1-1-4-1-3
1-1-1-3-4
1-1-3-1-4
1-1-4-2-1
1-1-1-4-2
1-1-1-2-4
1-1-2-1-4
1-1-4-3-1
1-1-1-4-3


C.7
1-1-1-3-3-(3)
1-1-3-1-3-(3)
1-1-3-3-1
1-3-1-3-1
C.8
1-1-2-2-2-(4)
1-2-1-2-2-(4)
1-2-2-1-2
1-4-1-2-2
1-1-2-2-4
1-2-1-2-4
1-2-2-1-4
1-1-2-4-2
1-2-1-4-2
1-1-4-2-2
C.9
1-1-2-2-3-(3)
1-1-2-3-2-(3)
1-1-2-3-3-(2)
1-1-3-2-2
1-1-3-2-3
1-1-3-3-2
1-2-1-2-3
1-2-1-3-2,
1-2-1-3-3
1-3-1-2-2
1-3-1-2-3
1-3-1-3-2
1-2-2-1-3
1-2-3-1-2
1-2-3-1-3
1-3-2-1-3
C.10
1-2-2-2-2-(3)
1-2-2-2-3
1-2-2-3-2
1-2-3-2-2
1-3-2-2-2
C.11
2-2-2-2-2-(2)
Table VI
C.1
1-1-1-1-1-1-(6)
C.2
1-1-1-1-1-2-(5)
1-1-1-1-2-1-(5)
1-1-1-2-1-1-(5)


C. 3
1-1-1-5-1-1-(2)
1-1-1-1-5-1-(2)
1-1-1-1-1-5-(2)
1-1-1-1-1-3-(4)
1 1 1 1 3 1 (4)
1 1 1 3 1 1 (4)
1 1 1 4 1 1 (3)
1 1 1 1 4 1 (3)
C. 4
1 1 1 1 1 4 (3)
1 1 1 1 2 2 (4)
1 1 1 2 1 2 (4)
1 1 2 1 1 2 (4)
1 1 1 2 4 1 (2)
1 1 1 1 2 4 (2)
1 1 1 2 2 1 (4)
1 1 2 1 2 1 (4)
1 1 2 1 4 1 (2)
1 1 1 2 1 4 (2)
1 1 2 2 1 1 (4)
1 1 4 1 2 1 (2)
1 1 2 1 1 4 (2)
1 1 1 4 1 2 (2)
1 1 1 4 2 1 (2)
1 1 1 1 4 2 (2)
C.5
1 1 1 1 2 3 (3)
1 1 1 3 2 1 (3)
1 1 3 1 2 1 (3)
1 1 2 1 3 1 (3)
1 1 1 2 1 3 (3)
1 1 1 1 3 2 (3)
1 1 1 3 1 2 (3)
1 1 3 1 1 2 (3)
1 1 1 2 3 1 (3)
1 1 1 1 3 3 (2)
1 1 1 3 1 3 (2)
1 1 3 1 1 3 (2)
C.6
1 1 1 3 3 1 (2)
1 1 3 1 3 1 (2)
1 1 2 1 1 3 (3)
1 1 1 2 2 2 (3)


1 1 2 1 2 2 (3)
1 1 2 2 3 1 (2)
1 1 1 2 2 3 (2)
1 1 2 2 1 2 (3)
1 2 1 2 1 2 (3)
1 1 2 1 2 3 (2)
1 1 2 3 1 2 (2)
1 1 2 3 2 1 (2)
1 1 1 2 3 2 (2)
1 1 2 2 2 1 (3)
1 2 1 2 2 1 (3)
1 1 2 2 1 3 (2)
1 2 1 3 1 2 (2)
1 2 1 2 1 3 (2)
1 1 2 1 3 2 (2)
1 1 3 1 2 2 (2)
1 1 3 2 1 2 (2)
1 1 3 2 2 1 (2)
C.7
1 1 1 3 2 2 (2)
1 1 2 2 2 2 (2)
1 2 1 2 2 2 (2)
1 2 2 1 2 2 (2)
Table VII
C.1
C.2
1 1 1 1 1 1 1 (5)
1 1 1 1 1 1 2 (4)
1 1 1 1 1 2 1 (4)
1 1 1 1 2 1 1 (4)
1 1 1 2 1 1 1 (4)
1 1 1 1 4 1 1 (2)
C.3
1 1 1 1 1 4 1 (2)
1 1 1 1 1 1 4 (2)
1 1 1 1 1 1 3 (3)
1 1 1 1 1 3 1 (3)
1 1 1 1 3 1 1 (3)
1 1 1 3 1 1 1 (3)

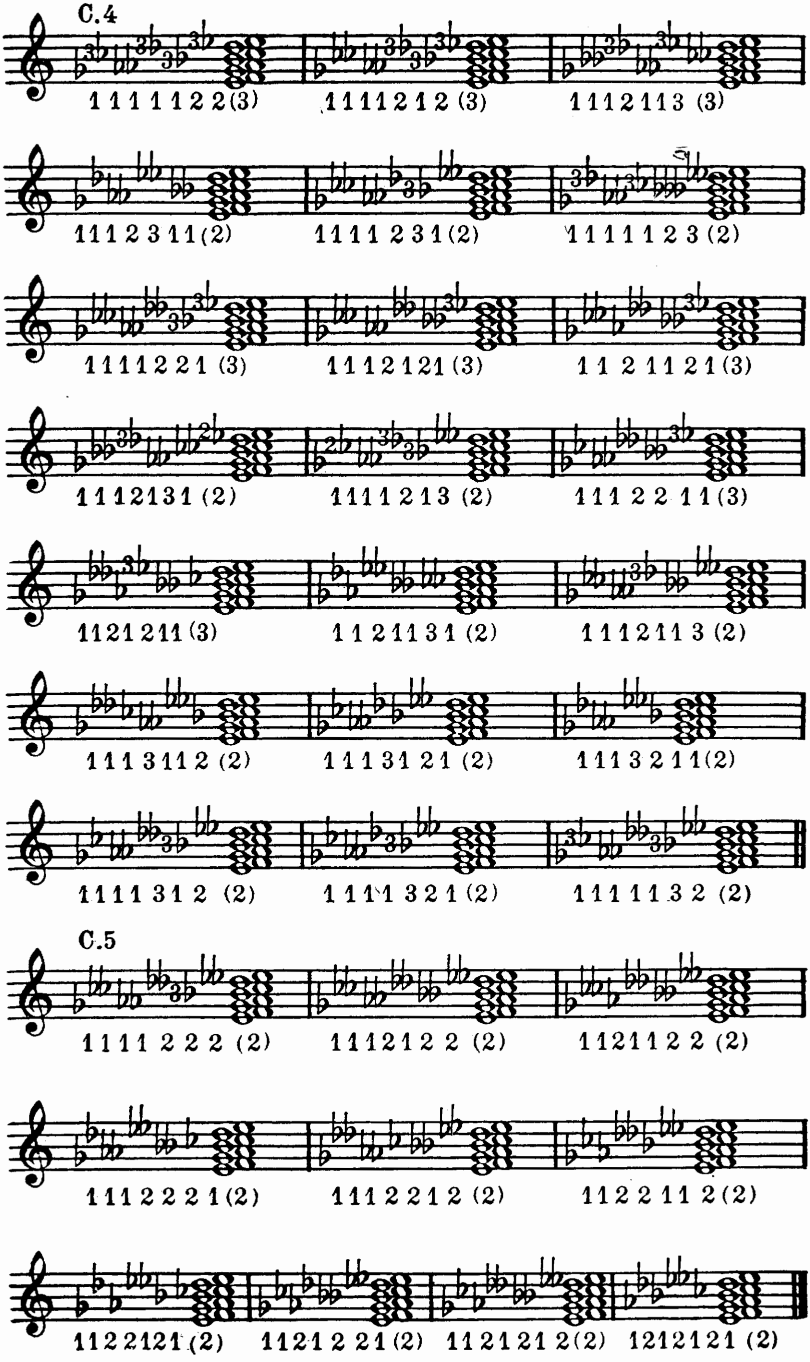

C.4
1 1 1 1 1 2 2 (3)
1 1 1 1 2 1 2 (3)
1 1 1 2 1 1 3 (3)
1 1 1 2 3 1 1 (2)
1 1 1 1 2 3 1 (2)
1 1 1 1 1 2 3 (2)
1 1 1 1 2 2 1 (3)
1 1 1 2 1 2 1 (3)
1 1 2 1 1 2 1 (3)
1 1 1 2 1 3 1 (2)
1 1 1 1 2 1 3 (2)
1 1 1 2 2 1 1 (3)
1 1 2 1 2 1 1 (3)
1 1 2 1 1 3 1 (2)
1 1 1 2 1 1 3 (2)
1 1 1 3 1 1 2 (2)
1 1 1 3 1 2 1 (2)
1 1 1 3 2 1 1 (2)
1 1 1 1 3 1 2 (2)
1 1 1 1 3 2 1 (2)
1 1 1 1 1 3 2 (2)
C.5
1 1 1 1 2 2 2 (2)
1 1 1 2 1 2 2 (2)
1 1 2 1 1 2 2 (2)
1 1 1 2 2 2 1 (2)
1 1 1 2 2 1 2 (2)
1 1 2 2 1 1 2 (2)
1 1 2 2 1 2 1 (2)
1 1 2 1 2 2 1 (2)
1 1 2 1 2 1 2 (2)
1 2 1 2 1 2 1 (2)

Table VIII

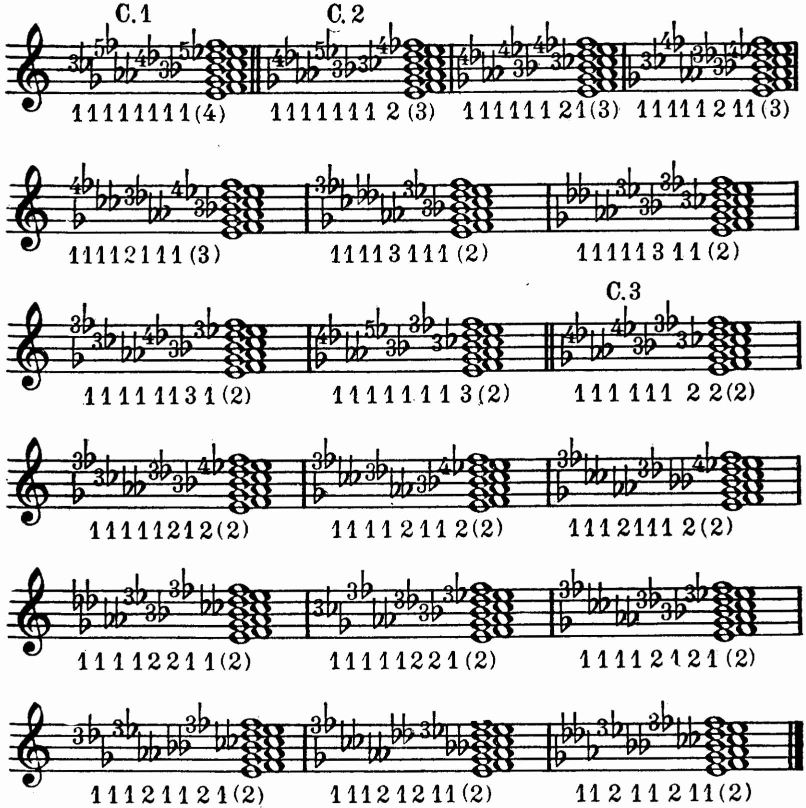


Table IX

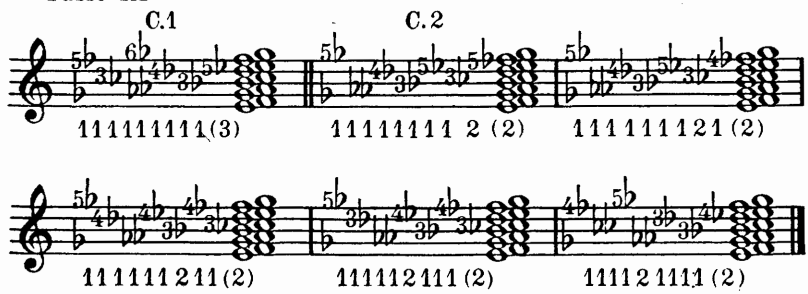


Table X

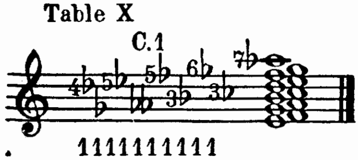
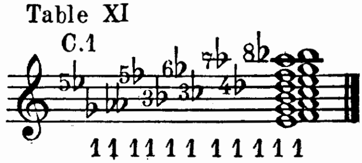


Table XI



NOTES ON THE NOTATION OF THE ABOVE HARMONIES.

1. R is represented in parentheses where it is included in the notation.
2. The degree most convenient for representation is chosen for the base note.
3. The Number Names under each respective harmony have their integers separated at first by dashes. Later these dashes are omitted.
4. When the number of sharps and flats becomes excessively great, it is written *n* ♭ or *n* ♯. Thus ♯♯♯♯ becomes 4♯.
5. There are other means of notating some of these denser harmonies. For example, it would be possible to employ two staffs, or double stems. Our present system of notation allows of no better methods.

Inversions of the Harmonies.

Many of these harmonies, especially those of many tones, may sound unesthetic in their fundamental form because of their dissonance, even to an ear trained to an appreciation of the most "ultra-modern" music. A conglomeration of slow beats caused by adjacent tones will, indeed, almost approach a common noise. However, such dissonance can be largely reduced in the same harmony when the tones of the fundamental position are scattered by octaves. Thus many harmonies, seemingly obscure in their fundamental position, become more appreciable to us by inverting them or spreading them out. The different forms and inversions of almost every harmony (made possible by the range of modern instruments) allow of the greatest variety of effects. The number of inversions and positions of most harmonies is astounding. Now, in making our rather superficial study of inversions, we will be obliged to use a few technical expressions; which are enumerated below.

A prime position of a harmony has been previously defined.

Any prime inversion of a fundamental[10] harmony will be known as *primary* inversion.

An inversion not prime, but containing no interval as great as the octave will be considered a *secondary* inversion.

An inversion containing one or more intervals exceeding the octave in magnitude will be known as a *tertiary* inversion.

An example of each type is given, respectively, below; the three inversions are in the same harmony. Although a form like (3) would, according to the chord system, be considered a fundamental position since the root (e) occupies the lowest position, we will

[10] That is, a harmony in fundaemntal position, according to definition.

find it more convenient to regard any position which is not prime, even though it have the root in lowest position, as an inversion.

Fundamental Pos. Primary Inv. Secondary Inv. Tertiary Inv.

In the following paragraphs, a few general principles are discussed in the form of propositions.

Proposition I. A harmony of n tones has $(n-1)$ inversions of the first degree.

Since a prime inversion can be formed with each tone as a lowest tone, and since $(n-1)$ tones are available as lowest tones (one tone being employed as the lowest tone for the fundamental) it follows that there are $(n-1)$ inversions possible.

In other words, an n-tone harmony has n prime positions.

Proposition II. The number of secondary inversions of a harmany of n tones is $(n!-n)$.

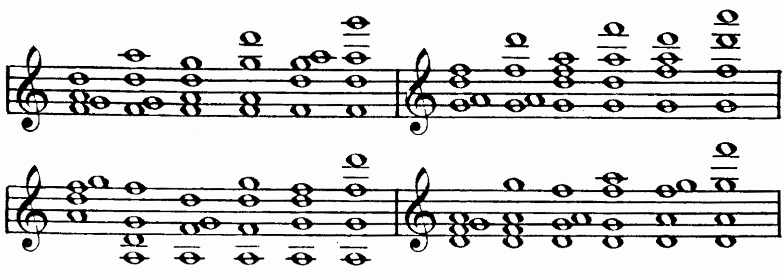
Let us experiment with two-, three- and four-tone harmonies, as follows, allowing no interval between adjacent notes to be as great as 12.

With two-tone harmonies we can form only two or 2! positions which conform to our limitations.

For example:

With three-tone harmonies only 6 or 3! such positions are possible; as for example:

With four-tone harmonies only 24 or 4! are possible:



With 5-tone harmonies 5! or 120 such inversions are possible, etc., etc. With n tones $n!$ positions of this type are possible. But since these positions in which the intervals are less than 12 include n prime positions, the secondary inversions number $(n!-n)$.

It is evident that the number of tertiary inversions is entirely dependent upon the range of the instrument employed to represent them.

By the *span* of a harmony is meant the interval of its extreme tones or the sum of its intervals. Thus the span of

5-9

is $(9+5)$ equals 14.[11]

Proposition III. In a harmony of n tones the sum of the spans of its prime positions is $(n-1)$ octaves. Let us take an example from a 4-tone harmony. Adding its prime positions we have

$$\begin{array}{cccccc} A+ & B+ & C & & \\ & B+ & C+ & R \\ A & + & C+ & R \\ A+ & B & + & R \\ \hline \end{array}$$

$$3A+3B+3C+3R=3(A+B+C+R)=3 \text{ octaves}$$
$$=(4-1) \text{ octaves.}$$

Progressions of Harmonies.

This subject interests us more from a speculative than a practical standpoint, since the possibilities in this direction are well-nigh unlimited, as will be shown. However, if the few suggestions that follow are carried out in limited form practical ends are attainable.

Any harmony may of course be preceded or followed by any other harmonies. Whether the progressions between harmonies sound abrupt or smooth depends partly upon the harmonies in question, partly upon the positions chosen, and partly upon the degree of broad appreciation to which we have been trained. However, smoothness or abruptness of progression does not concern us here, for either may be more desirable according to the character of a composition. The inclination of a composer with ideas certainly

[11] In finding the span of a harmony, R is evidently not included to make the sum 12, since the span of all harmonies would consequently be multiples of 12.

deserves more consideration than the ever weakening law of theorists.

Let us now calculate the number of progressions of two successive harmonies which can be made with a given number of harmonies, let us say n harmonies. One of these n harmonies placed upon one degree of the chromatic scale can progress to the same harmony placed on 11 other degrees. The same harmony can form 12 progressions with any of the other harmonies given, since any other harmony can be placed on 12 different degrees. And since there are $(n-1)$ such other harmonies, the first harmony can form $(n-1)12$ progressions with the remaining harmonies. Thus the total number of progressions possible between a single harmony and the remaining harmonies is: $11+12\cdot(n-1)$.

But as many progressions are possible with each of the $(n-1)$ remaining harmonies. Hence the total number of progressions of two successive harmonies possible with n harmonies is:

$$n[11+12(n-1)]=(12n-1)n \text{ or } 12n^2-n.$$

Thus, with only 2 harmonies we can form $(24-1)2$, or 46 progressions. With 5 harmonies (which is the limit of vocabulary with many persons, and in which may be included the major triad, minor triad, dominant sept, supertonic sept and leading-tone sept), $(60-1)5$ or 295 progressions of only two successive ones are possible. With the 350 existing harmonies, the possibilities of progression of two at a time are: $(350\times 12-1)\,350=1{,}469{,}650$.

The number of possibilities of progressions of three at a time will be $(12n^2-n)12n$, since each progression of two harmonies may be followed by one of n harmonies placed on any one of 12 different degrees of the scale. Thus with 350 harmonies 6,172,530,000 progressions of three are possible.

The general formula expressing the number of progressions possible is:

$$(12n^2-n)(12n)^{s-2}$$

where n is the number of harmonies among which the progressions are to be made, while s is the number of harmonies to be used at a time in a progression.

As mentioned before, the enormous figures just given mean little practically, yet they serve to emphasize the fact that variety is not only desirable but possible; and this is only variety of one kind, harmonic variety.

Melody, rhythm and form are quite as variable as harmony, and the variability of music is measured as the product of these respective elements of it and is therefore quite beyond the bounds of comprehension. Formerly I shared the foolish and common fear that as more music is written the possibilities of future invention narrow. I actually felt that the field for contemporary composition is narrower than it was a century ago. To-day it seems to me that every great musical work enlarges the field of the future.

"And myriad strains are there since the beginning still waiting for manifestation."[12]

ERNST LECHER BACON.

CHICAGO, ILLINOIS.

[12] Busoni, *A New Esthetic of Music.*